How to... # Compute limits of rational sequences

Given: A sequence with rational elements, i.e. a sequence of the form

$$a_n = \frac{\alpha_p\, n^p + \alpha_{p-1}\, n^{p-1} + \cdots + \alpha_2\, n^2 + \alpha_1\, n + \alpha_0}{\beta_q\, n^q + \beta_{q-1}\, n^{q-1} + \cdots + \beta_2\, n^2 + \beta_1\, n + \beta_0}$$

where $p, q \in \mathbb{N}$ are some numbers (the highest exponents in the numerator and the denominator, respectively) and $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_q$ are some real coefficients.

Wanted: The limit

$$a := \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\alpha_p\, n^p + \alpha_{p-1}\, n^{p-1} + \cdots + \alpha_2\, n^2 + \alpha_1\, n + \alpha_0}{\beta_q\, n^q + \beta_{q-1}\, n^{q-1} + \cdots + \beta_2\, n^2 + \beta_1\, n + \beta_0}$$

Example

Consider the following sequences.

$$a_n := \frac{4n^3 + 3n^2 - 2n + 42}{6n^3 - 2n^2 + n}$$
$$b_n := \frac{2n^2 + n}{n^4 + n^3 - 2n^2 - 1}$$
$$c_n := \frac{n^2 - n - 5}{n + 1}$$

1 Find the highest exponent

Determine the highest exponent $r := \max\{p, q\}$ in both numerator and denominator of the sequence.

For a_n, the highest exponent is $r = 3$ (in both numerator and denominator we have a n^3-term).
For b_n, the highest exponent is $r = 4$ (there is a n^4 term in the denominator).
For c_n, the highest exponent is $r = 3$ (there is a n^2 term in the numerator).

2 Factor out the highest-exponent-term

Factor out the term n^r (i.e. the n term with the highest exponent) in both numerator and denominator. Then cancel the n^r term.

Factoring out $n^3/n^4/n^2$ in the sequences $a_n/b_n/c_n$ yields

$$a_n = \frac{n^3 \cdot \left(4 + \frac{3}{n} - \frac{2}{n^2} + \frac{42}{n^3}\right)}{n^3 \cdot \left(6 - \frac{2}{n} + \frac{1}{n^2}\right)} = \frac{4 + \frac{3}{n} - \frac{2}{n^2} + \frac{42}{n^3}}{6 - \frac{2}{n} + \frac{1}{n^2}}$$

$$b_n = \frac{n^4 \cdot \left(\frac{2}{n^2} + \frac{1}{n^3}\right)}{n^4 \cdot \left(1 + \frac{1}{n} - \frac{2}{n^2} - \frac{1}{n^4}\right)} = \frac{\frac{2}{n^2} + \frac{1}{n^3}}{1 + \frac{1}{n} - \frac{2}{n^2} - \frac{1}{n^4}}$$

$$c_n = \frac{n^2 \cdot \left(1 - \frac{1}{n} - \frac{5}{n}\right)}{n^2 \cdot \left(\frac{1}{n} + \frac{1}{n}\right)} = \frac{1 - \frac{1}{n} - \frac{5}{n}}{\frac{1}{n} + \frac{1}{n^2}}$$

3 Use the rules for limits

Use the following rules for limits

$$\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c \cdot a_n = c \cdot \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{\lim_{n\to\infty} a_n}$$

$$\lim_{n\to\infty} \frac{1}{n^k} = 0 \qquad \text{for any } k > 0$$

to compute the limit of numerator and denominator separately.
If the denominator converges to zero, while the numerator converges to a real, non-zero number, the whole sequence is divergent.

We can compute the limits of a_n and b_n easily, as the denominator converges to a non-zero number in both cases:

$$\lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty} 4 + \frac{3}{n} - \frac{2}{n^2} + \frac{42}{n^3}}{\lim_{n\to\infty} 6 - \frac{2}{n} + \frac{1}{n^2}} = \frac{4+0+0+0}{6-0+0} = \frac{2}{3}$$

$$\lim_{n\to\infty} b_n = \frac{\lim_{n\to\infty} \frac{2}{n^2} + \frac{1}{n^3}}{\lim_{n\to\infty} 1 + \frac{1}{n} - \frac{2}{n^2} - \frac{1}{n^4}} = \frac{0+0}{1+0-0-0} = 0$$

For the limit of the sequence c_n, we observe that the numerator of c_n converges to $\lim_{n\to\infty} 1 - \frac{1}{n} - \frac{5}{n} = 1$ while the denominator of c_n converges to zero ($\lim_{n\to\infty} \frac{1}{n} + \frac{1}{n^2} = 0$). Thus, the whole sequence diverges (in this case to $+\infty$). So

$$\lim_{n\to\infty} c_n = +\infty$$